Variable Separation Approach for a Differential-difference Asymmetric Nizhnik-Novikov-Veselov Equation

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The multi-linear variable separation approach is applied to a differential-difference asymmetric Nizhnik-Novikov-Veselov equation. It is found that the solution formula ANNV equation is rightly the semi-discrete form of the continuous one which describes some types of special solutions for many (2+1)-dimensional continuous systems. Moreover, it is different but similar to that of a special differential-difference Toda system. Thus abundant semi-discrete localized coherent structures of the ANNV equation are easily constructed by appropriately selecting the arbitrary functions appearing in the final solution formula. A concrete method to construct multiple localized discrete excitations with and without completely elastic interaction properties are discussed. It is found that for some types of bounded solitoff modes, they can exchange solitoffs, move in new different directions, extend the lengths of solitoffs, etc. after finishing their interactions. – PACS numbers: 0230, 0220, 0540

Key words: Multi-linear Variable Separation Approach; Differential-difference ANNV Equation; Semi-discrete Localized Coherent Structures.

1. Introduction

Recently, several kinds of "variable separation" approaches have been established, say, the classical method, the differential Stäckel matrix approach [1], the geometric method [2], the ansatz-based methods [2, 3], the functional variable separation approach (FVSA) [4], the derivative-dependent functional variable separation approach (DDFVSA) [5], the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints) [6] and the multi-linear variable separation approach (MLVSA). Among these variable separation approaches, the MLVSA is considered as the most powerful method to find special types of exact solutions for (2+1)-dimensional nonlinear systems.

The MLVSA is basically based on Hirota's bilinear direct method [7] and its multi-linear extensions [8]. It has been successively applied to a diversity of (2+1)-dimensional systems including the Davey-Stewartson (DS) system, the Nizhnik-Novikov-Veselov (NNV) equation, the modified NNV equation, the asymmetric NNV equation, the asymmetric DS equation, the dispersive long wave equation, the Broer-Kaup-Kupershmidt (BKK) system, the higher order BKK system, the nonintegrable (2+1)-dimensional KdV equation, the (2+1)-dimensional Burgers equation, the long wave-short wave interaction model, the Maccari system, the general (N+M)-component AKNS system, the (2+1)-dimensional sine-Gordon model, etc. [9 – 17]. Especially, in [13] it is extended to a differential-difference system, a special Toda equation (SDDTE)

$$Q(n)_{yt} = \exp[Q(n+1)-Q(n)][Q(n+1)+Q(n)]_y - \exp[Q(n)-Q(n-1)][Q(n)+Q(n-1)]_y, \quad (1)$$

where $Q(n) \equiv Q(n,\ y,\ t)$ is a function of the discrete variable n and the continuous ones $\{y,\ t\}$.

In the continuous case [11], the formula

$$U \equiv \frac{2dq_y p_x}{(a_0+a_1p+a_2q+a_3pq)^2},\ d \equiv (a_1a_2-a_0a_3) \quad (2)$$

is derived to describe some special solutions for some suitable physical quantities of all models mentioned above. Hereafter the models are called MLVSA solvable models. In (2), $a_0,\ a_1,\ a_2$ and a_3 are arbitrary constants, p is an arbitrary function of $\{x,\ t\}$ for all of the known MLVSA solvable models, while q of (2) may be an arbitrary function of $\{y,\ t\}$ for some, or an arbitrary solution of a special equation (say, the Riccati equation or the diffusion equation) for some others. Because some arbitrary characteristics, lower dimensional functions (like p), have been included in the formula (2), by selecting them appropriately, abundant localized structures like the multiple solitoffs, dromions, lumps, breathers, instantons, peakons, compactons, foldons, ghostons, ring solitons, chaotic and fractal patterns have been found [11].

For the SDDTE (1), the semi-discrete form of the quantity (2) reads

$$u = U(n) \equiv \frac{2dq_y[p(n+1)-p(n)]}{[a_0+a_1p(n)+a_2q+a_3qp(n)][a_0+a_1p(n+1)+a_2q+a_3qp(n+1)]}. \tag{3}$$

In this paper we are interested in the important question: Can the MLVSA be extended to solve other nonlinear DDEs and whether the semi-discrete form (3) is the same for other MLVSA solvable DDEs? In Sects. 2 and 3, the bilinear variable separation approach (BLVSA), a special case of the MLVSA, is applied to a differential-difference ANNV equation (DDANNVE). In Sect. 4, it is proved that the semi-discrete form (2) for a suitable quantity of the DDANNVE is similar to but different from (3). The abundant semi-discrete localized excitations are constructed and depicted also in Sect. 4. The last section contains the conclusions and discussions.

2. DDANNVE and its Generalized Bilinear Form

One of the integrable DDANNVEs can be written as (DDANNVE)

$$v_x(n+1)+v_x(n) = w(n+1)-w(n), \tag{4}$$

$$\begin{aligned} &v_t(n)+3v_x(n)v^2(n)+v_{xxx}(n)\\ &\quad+3w(n)v_x(n)+3w_x(n)v(n)=0 \end{aligned} \tag{5}$$

where $v(n)\equiv v(n,\ x,\ t)$ and $w(n)\equiv w(n,\ x,\ t)$ are functions of the discrete variable n and the continuous variables $\{x,\ t\}$.

In the study of modern soliton theory, to find the integrable discretizations of continuous integrable physical systems is one of the most important hot topic because of the rapid development of computer science. Though we have not yet found a direct application of the DDANNVEs (4) and (5), the model is still interesting because its continuous version is one of the important (2+1)-dimensional integrable models and possesses possible physical applications [18].

The DDANNVEs (4) and (5) can be transformed into the famous ANNV equation ([10]) by the continuous analogue of the equations. Setting

$$\begin{aligned} v(n,\ x,\ t) &= -\frac{1}{2}\varepsilon V(\varepsilon n,x,t) = -\frac{1}{2}\varepsilon V(y,x,t),\\ w(n,x,t) &= -W(\varepsilon n,x,t) = -W(y,x,t), \end{aligned} \tag{6}$$

we have

$$\begin{aligned} v(n+1,x,t) &= -\frac{1}{2}\varepsilon V(\varepsilon n+\varepsilon,x,t)\\ &= -\frac{1}{2}\varepsilon V(y+\varepsilon,x,t)\\ &= -\frac{1}{2}\varepsilon V(y,x,t)+O(\varepsilon^2), \end{aligned} \tag{7}$$

$$\begin{aligned} w(n+1,x,t) &= -W(\varepsilon n+\varepsilon,x,t) = -W(y+\varepsilon,x,t)\\ &= -(W(y,x,t)+\varepsilon W_y(y,x,t))+O(\varepsilon^2). \end{aligned} \tag{8}$$

Substituting these expressions into (4) and (5) and neglecting the higher order terms of ε, we have

$$V_x = W_y, \tag{9}$$

$$V_t+V_{xxx}-3WV_x-3W_xV=0 \tag{10}$$

being just the well known ANNV equation. The continuous ANNV equation (10) can be considered as a model for an incompressible fluid, where u and v are the components of the dimensionless velocity [18]. The spectral transformation for this system has been investigated in [19] and [20]. This system has been considered also in [21] as a generalization to (2+1) dimensions of the results from Hirota and Satsuma [22]. The nonclassical symmetries, Painlevé property, and similarity solutions of the system have been studied by Clarkson and Mansfield [23]. The ANNV system can also be obtained from a special inner parameter dependent symmetry reduction from the well known Kadomtsev-Petviashvili equation [24] which has wide applications in nonlinear physics.

To get exact solutions of the DDANNVE via BLVSA, we take the following dependent variable transformation:

$$v(n)=\left(\ln\frac{f(n+1)}{f(n)}\right)_x+v_0,\qquad w(n)=(\ln(f(n+1)f(n)))_{xx}+w_0. \tag{11}$$

where $\{v_0,w_0\}$ is an arbitrary seed solution of the DDANNVE. Similar to the continuous case, we take the seed solution as

$$v_0=v_0(t),\qquad w_0=w_0(x,t), \tag{12}$$

where v_0 is an arbitrary function of t, and w_0 is an arbitrary function of x and t.

Substituting the dependent variable transformation (11) into (5) yields a bilinear form of the DDANNVE,

$$\begin{aligned}&D_t f(n+1)\cdot f(n)+D_x^3 f(n+1)\cdot f(n)\\&+3v_0D_x^2 f(n+1)\cdot f(n)+3v_0^2D_x f(n+1)\cdot f(n)\\&+3w_0D_x f(n+1)\cdot f(n)+c(n,t)f(n+1)\cdot f(n)=0,\end{aligned} \tag{13}$$

where $c(n,t)$ is a function of $\{n,t\}$. Hirota's bilinear differential operator D_x^m is defined by

$$D_x^m a\cdot b\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m a(x,t)b(x',t')\bigg|_{x=x',t=t'}.$$

3. Variable Separation Solutions of the DDANNVE

In order to find some exact solutions of (13), the next important step is to take a suitable solution ansatz for the function $f(n)$. Similar to the continuous cases [9–11], we look for the solutions of (13) in the form

$$f(n)=a_0+a_1p(x,t)+a_2q(n,t)+a_3p(x,t)q(n,t), \tag{14}$$

where a_0, a_1, a_2 and a_3 are arbitrary constants and the variable separated functions $p(x,\ t)\equiv p$ and $q(n,\ t)\equiv q(n)$ are only functions of $\{x,\ t\}$ and $\{n,\ t\}$, respectively. The Equation (14) looks like Hirota's two-soliton form when p and $q(n)$ are exponential functions. Substituting the ansatz (14) into (13), we have ($P\equiv a_2+a_3p$, $Q_n\equiv a_1+a_3q(n)$)

$$\begin{aligned}&d[q(n)-q(n+1)][p_t+p_{xxx}+3w_0p_x+3v_0^2p_x]\\&+P\left(f_n\frac{\partial q(n+1)}{\partial t}-f_{n+1}\frac{\partial q(n)}{\partial t}\right)\\&+c(n,t)f_nf_{n+1}-6v_0Q_{n+1}p_x^2\\&+3v_0\big[2a_1^2p+2a_1a_3pq(n+1)+2a_1a_3pq(n)\\&\quad+a_1a_2q(n)+2a_0a_1+a_1a_2q(n+1)\\&\quad+2a_3^2q(n+1)q(n)p+a_0a_3q(n)\\&\quad+2a_2a_3q(n+1)q(n)+a_0a_3q(n+1)\big]p_{xx}=0.\end{aligned} \tag{15}$$

Because $q(n)$ is x-independent and p is n-independent, (15) can be separated into the two equations

$$p_t=-p_{xxx}-3p_xw_0+\beta_2p^2+\beta_1p+\beta_0, \tag{16}$$

$$\frac{\partial q(n)}{\partial t}=k_2q(n)^2+k_1q(n)+k_0 \tag{17}$$

when β_0, β_1, β_2 and $c(n,t)$ are selected as

$$\begin{aligned}\beta_0&=-\frac{(-a_0a_2k_1+a_0^2k_2+a_2^2k_0)}{(a_3a_0-a_1a_2)},\\ \beta_1&=-\frac{(a_3a_0k_1+a_1a_2k_1-2a_0a_1k_2-2a_2a_3k_0)}{(a_3a_0-a_1a_2)},\\ \beta_2&=\frac{(a_1^2k_2-a_1a_3k_1+a_3^2k_0)}{(a_3a_0-a_1a_2)},\end{aligned} \tag{18}$$

$$c(n,t)=k_2(q(n)-q(n+1)), \tag{19}$$

with $k_0\equiv k_0(t)$, $k_1\equiv k_1(t)$ and $k_2\equiv k_2(t)$ being arbitrary functions of t.

In principle, as long as the arbitrary functions k_0, k_1, k_2 and w_0 are fixed, we can obtain the corresponding special solutions of (16) and (17), and then the special solutions of the DDANNVE (4) and (5). However, it is still very difficult to solve (16) for fixed nonzero w_0. Fortunately, just as in the continuous cases [9–17], the arbitrariness of the function w_0 allows us treat the problem alternatively: Consider the function p as an arbitrary function of the variables x and t, and fix the function w_0 by

$$w_0=(-3p_x)^{-1}(p_t+p_{xxx}-\beta_2p^2-\beta_1p-\beta_0). \tag{20}$$

It should be pointed out that the Riccati equation (17) is totally the same as that in the continuous ANNV equation [11]. To find some special solutions of (17), one may select the arbitrary functions appropriately. Here we list two special selections.

1. If we write $k_0,\ k_1$ and k_2 as

$$k_0 = \frac{(A_1A_{2t} - A_2A_{1t} - A_2^2A_{3t})}{A_1},\quad k_1 = \frac{(A_{1t} + 2A_2A_{3t})}{A_1},\quad k_2 = -\frac{A_{3t}}{A_1} \tag{21}$$

with $A_1 \equiv A_1(t),\ A_2 \equiv A_2(t)$ and $A_3 \equiv A_3(t)$ being arbitrary functions of t, then the general solution of (17) with (21) reads

$$q(n) = \frac{A_1}{A_3 + F_1(n)} + A_2, \tag{22}$$

where $F_1 \equiv F_1(n)$ is an arbitrary function of n. By means of (21), we can rewrite (18) as

$$\begin{aligned}\beta_0 &= -\frac{a_2(a_2A_2 + a_0)A_{1t}}{(a_3a_0 - a_1a_2)A_1} - \frac{(a_2A_2 + a_0)^2A_{3t}}{(a_3a_0 - a_1a_2)A_1} + \frac{a_2^2A_{2t}}{(a_3a_0 - a_1a_2)},\\ \beta_1 &= -\frac{(2a_3a_2A_2 + a_1a_2 + a_0a_3)A_{1t}}{(a_3a_0 - a_1a_2)A_1} - \frac{2(a_2A_2 + a_0)(a_3A_2 + a_1)A_{3t}}{(a_3a_0 - a_1a_2)A_1} + \frac{2a_2a_3A_{2t}}{(a_3a_0 - a_1a_2)},\\ \beta_2 &= -\frac{a_3(a_3A_2 + a_1)A_{1t}}{(a_3a_0 - a_1a_2)A_1} - \frac{(a_3A_2 + a_1)^2A_{3t}}{(a_3a_0 - a_1a_2)A_1} + \frac{a_3^2A_{2t}}{(a_3a_0 - a_1a_2)}.\end{aligned} \tag{23}$$

2.) If we select $k_0,\ k_1$ and k_2 as

$$k_0 = -\frac{(-B_{1t}B_0 + B_1^2B_{2t} - B_0^2B_{2t} + B_1B_{0t})}{B_1},\quad k_1 = -\frac{(-B_{1t} + 2B_0B_{2t})}{B_1},\quad k_2 = \frac{B_{2t}}{B_1} \tag{24}$$

with $B_0 \equiv B_0(t),\ B_1 \equiv B_1(t)$ and $B_2 \equiv B_2(t)$ being arbitrary functions of t, then the general solution of (17) with (24) can be written as

$$q(n) = B_1 \tanh(B_2 + F_2(n)) + B_0, \tag{25}$$

with $F_2 \equiv F_2(n)$ being an arbitrary function of n, while the functions $\beta_0,\ \beta_1$ and β_2 should be determined by

$$\begin{aligned}\beta_0 &= -\frac{a_2(a_2B_0 - a_0)B_{1t}}{(a_1a_2 - a_3a_0)B_1} + \frac{a_2^2B_{0t}}{(a_1a_2 - a_3a_0)} - \frac{(a_0^2 - a_2^2B_1^2 + a_2^2B_0^2 + 2a_0a_2B_0)B_{2t}}{(a_1a_2 - a_3a_0)B_1},\\ \beta_1 &= \frac{(-2a_3a_2B_0 + a_1a_2 + a_0a_3)B_{1t}}{(a_1a_2 - a_3a_0)B_1} + \frac{2a_2a_3B_{0t}}{a_1a_2 - a_3a_0} + \frac{(2a_2a_3B_1^2 - 2a_2a_3B_0^2 - 2a_3a_0B_0 - 2a_1a_2B_0 - 2a_0a_1)B_{2t}}{(a_1a_2 - a_3a_0)B_1},\\ \beta_2 &= -\frac{(a_3^2B_0 - a_1a_3)B_{1t}}{(a_1a_2 - a_3a_0)B_1} - \frac{(-a_3^2B_1^2 + a_3^2B_0^2 + a_1^2 + 2a_1a_3B_0)B_{2t}}{(a_1a_2 - a_3a_0)B_1} + \frac{a_3^2B_{0t}}{(a_1a_2 - a_3a_0)}.\end{aligned} \tag{26}$$

4. Multiple Localized Excitations with and without Complete Elastic Interaction Property

Substituting all the results obtained in the last section into (11) arrives at many kinds of exact solutions for the fields v_n and w_n of the DDANNVE. In continuous cases, for every MLVSA solvable system listed in [11] there exits a quantity whose special solutions can be expressed by (2). Naturally it is important to ask: Is there a suitable quantity for the DDANNVE such that it can be described by a suitable semi-discrete form of (2)?

By substituting the result of the last section into (11), it is straightforward to see that

$$v(n) = -\frac{1}{2}U_1(n) \equiv \frac{-dp_x(q(n+1)-q(n))}{(a_0+a_1p+a_2q(n)+a_3pq(n))(a_0+a_1p+a_2q(n+1)+a_3pq(n+1))}, \quad (27)$$

$$w(n) = w_0 + \frac{[(2dQ_nP-2Q_n^2P^2-d^2)Q_{n+1}^2+dQ_n^2(2PQ_{n+1}-d)]p_x^2}{a_3^2(a_0+a_1p+a_2q(n)+a_3pq(n))^2(a_0+a_1p+a_2q(n+1)+a_3pq(n+1))^2} + \frac{[2Q_nQ_{n+1}P-d(Q_n+Q_{n+1})]p_{xx}}{a_3(a_0+a_1p+a_2q(n)+a_3pq(n))(a_0+a_1p+a_2q(n+1)+a_3pq(n+1))}. \quad (28)$$

The function $U_1(n)$ defined in (27) is another suitable semi-discrete form of the continuous quantity U given by (2). It should be noticed that, though the semi-discrete form (27) for the DDANNVE is quite similar to (3) obtained for the SDDTE, two discrete forms are not completely same. For the discrete form (3), which is responsible for the SDDTE, the arbitrary function is related to the discrete space variable, and the other function related to the continuous space variable should be a solution of the Riccati equation. However, for the discrete form (27), which is responsible for the DDANNVE, the arbitrary function is related to the continuous space variable, and the other one, related to the discrete space variable, is a solution of the Riccati equation.

Under the limiting procedure (6), $\varepsilon n \to y$, we find that the error between the continuous quantity U expressed by (2) and U_1/ε given by (27) reads

$$\frac{U_1}{\varepsilon} - U = \Big(\frac{dp_xq_{yy}}{(a_0+a_1p+a_2q+a_3pq)^2} - 2\frac{dp_xq_y^2}{(a_0+a_1p+a_2q+a_3pq)^3}\Big)\varepsilon + O(\varepsilon^2). \quad (29)$$

Starting from the semi-discrete quantity u expressed by (27), we can obtain abundant semi-discrete localized excitations for the DDANNVE by selecting the arbitrary functions suitably.

Detailed studies show that the semi-discrete localized structures for u (such as the multiple solitoffs (half-infinite straight line solitons or straight line solitons with finite length), dromions, lumps, breathers, instantons, peakons, compactons, foldons, ghostons, ring solitons and chaotic and fractal patterns) are quite similar to the continuous ones which have been discussed in [11] for the continuous models, and some examples of special discrete single localized excitations related to (3) have been discussed for the SDDTE [13]. So, here we will not discuss all the possible localized excitations but give some special examples of the DDANNVE similar to those of the SDDTE.

Example 1. Resonant semi-discrete dromions and solitoff solutions:

If we restrict the functions $q(n)$ and p of (27) to

$$q(n) = \sum_{i=1}^{N}\sum_{j=1}^{N_1}\exp(c_in+\omega_jt+\tau_{0i}), \quad (30)$$

$$p = \sum_{i=1}^{M}\exp(K_ix+\Omega_it+\xi_{0i}), \quad (31)$$

where $\tau_{0i}, \xi_{0i}, c_i,\ \omega_j,\ \Omega_j$ and K_i are arbitrary constants, and M, N and N_1 are arbitrary positive integers, then we have resonant semi-discrete dromion solutions or semi-discrete multiple solitoff solutions. The selection (30) is related to selections of the functions $A_i\ (i=1,\ 2,\ 3)$ and F_1 in (22), which are

$$A_3 = A_2 = 0, \quad (32)$$

$$A_1 = \sum_{j=1}^{N_1}\exp(\omega_jt), \quad (33)$$

$$F_1 = 1/\sum_{i=1}^{N}\exp(c_in+\tau_{0i}), \quad (34)$$

and $k_i\ (i=0,\ 1,\ 2)$ are given by (21) with (32) and (33).

In Figs. 1 – 4, four typical evolution structures caused by the resonant effects of four straight-line semi-discrete soliton solutions are plotted.

Figure 1 shows the evolution structure of a first type of single resonant semi-discrete dromion solutions

Fig. 1a

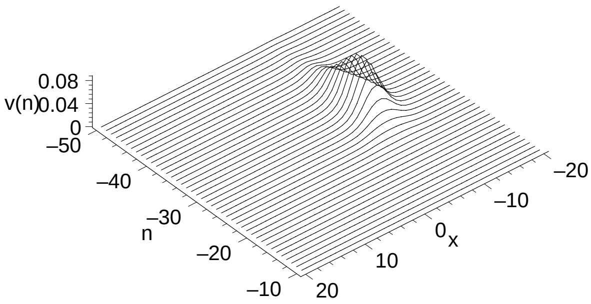

Fig. 1b

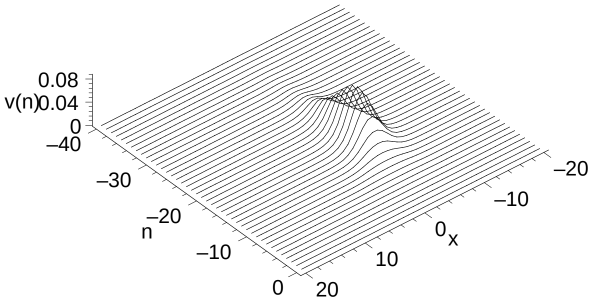

Fig. 1c

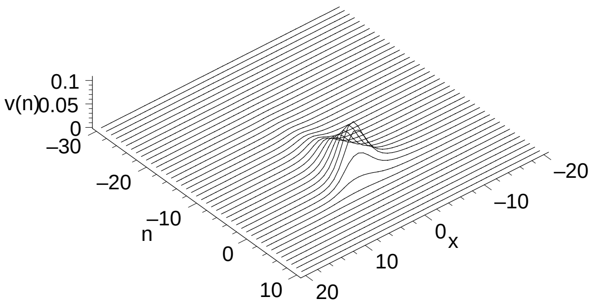

Fig. 1d

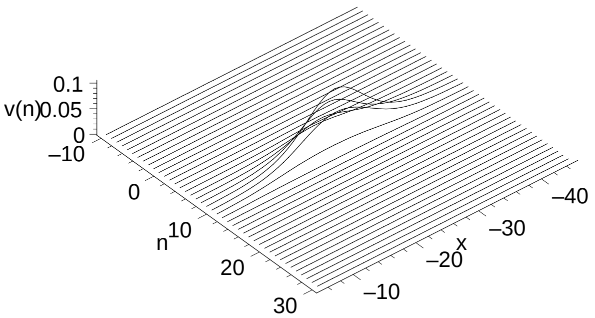

Fig. 1. A special semi-discrete resonant dromion of the DDANNVE for the field $u(n)$ expressed by (27) with (30)–(31). and (35)–(36) at the times a: $t = -10$, b: $t = -5$, c: $t = 0$, and d: $t = 6$, respectively.

Fig. 2a

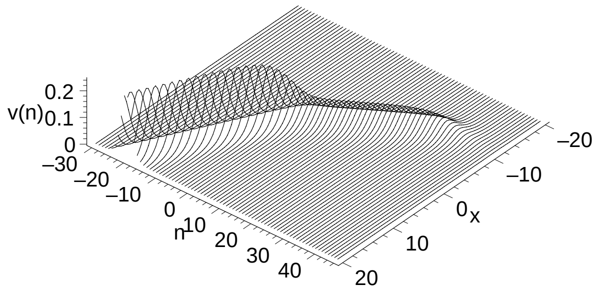

Fig. 2b

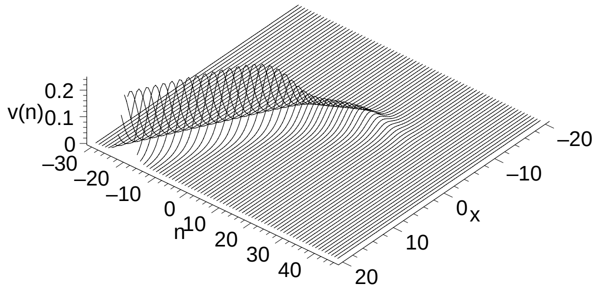

Fig. 2c

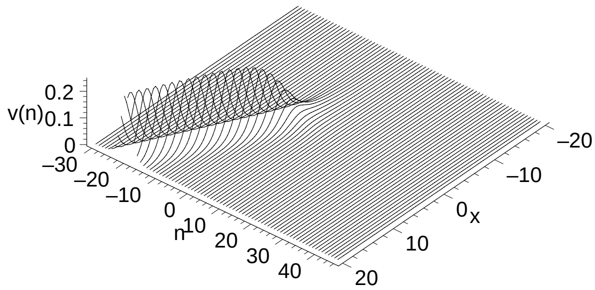

Fig. 2d

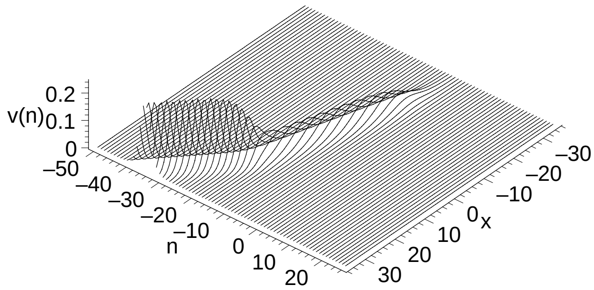

Fig. 2. A special resonant multi-solitoff solution for the field $v(n)$ expressed by (27) with (30)–(36) and (37) at the times a: $t = -10$, b: $t = -5$, c: $t = 0$ and d: $t = 6$ respectively.

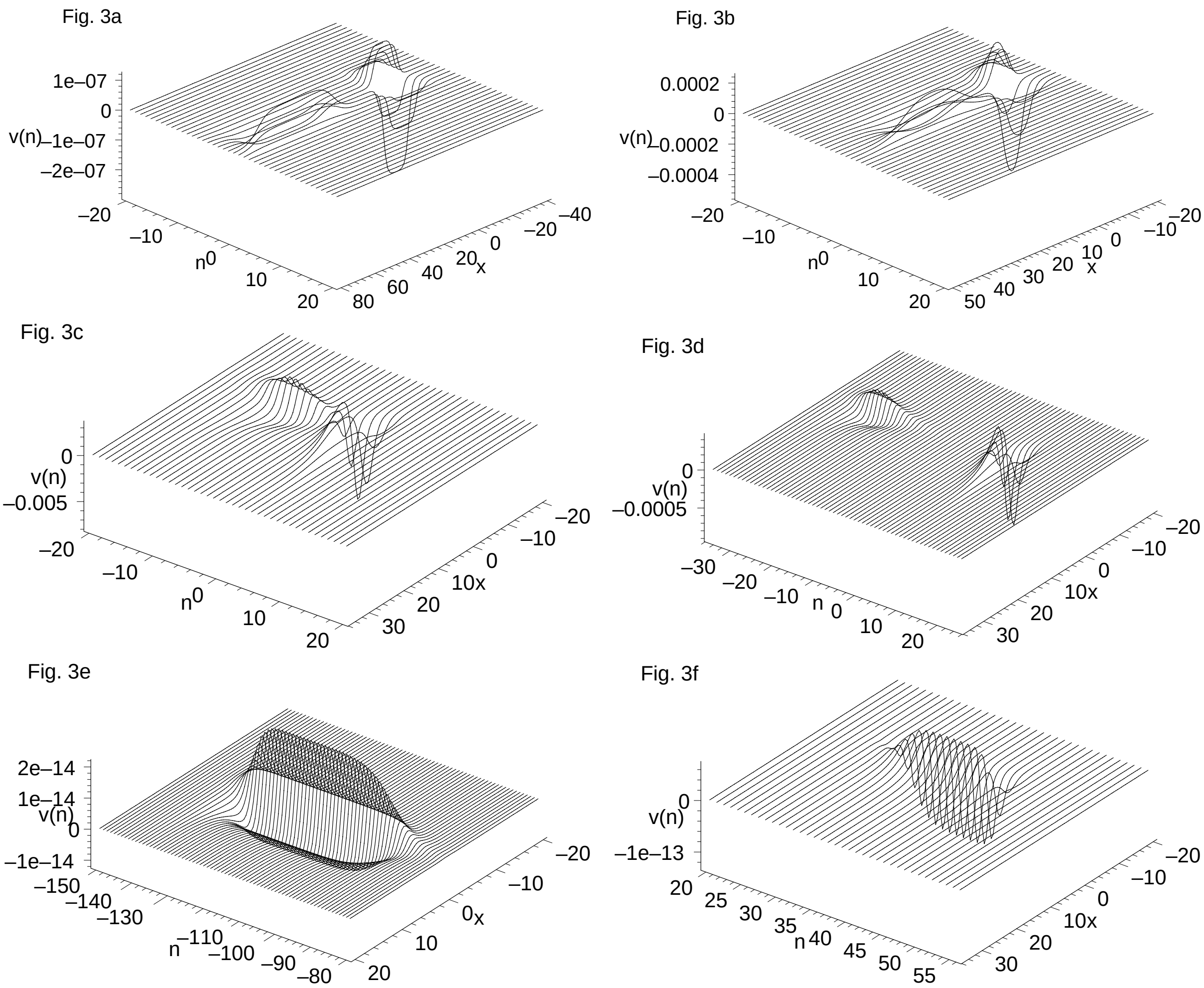

Fig. 3. The evolution plots of four semi-discrete resonant solitoffs of the DDANNVE for the field $v(n)$ expressed by (27) with (30)–(36) and (38) at the times a: $t=-8$, b: $t=-4$, c: $t=0$ and d: $t=4$. The figures (e) and (f) are the plots of the left and right solitoff pair structures related to the figure d at time $t=28$.

expressed by (27) with (30), (31),

$$M=N=2,\ N_1=c_1=K_1=-\omega_1=\Omega_2=-\Omega_1=1,$$

$$c_2=K_2=\frac{1}{3},\ a_0=a_3=3,\ a_1=1/2,\ a_2=1, \qquad (35)$$

and

$$\tau_{01}=\xi_{01}=\tau_{02}=\xi_{02}=0, \qquad (36)$$

at times $t=-10,\ -5,\ 0$ and 10 respectively. From the figures of Fig. 1 we know that for a *single* dromion there is also a special "interaction" time (which is determined by four invisible straight travelling line solitons), $t=0$, in this case. Before the special interaction time, the resonant dromion possesses a small amplitude. After the interaction, the amplitude of the resonant dromion becomes larger. The shape of the dromion is also changed after that interaction time.

Figure 2 is a plot of the evolution of a two-resonant semi-discrete solitoff solution shown by (27) with (30), (31), (36) and

$$M=N=2,\ N_1=-c_1=K_1=-\omega_1=\Omega_2=-\Omega_1=1,$$

$$-c_2=K_2=\frac{1}{3},\ a_0=1,\ a_1=a_2=3,\ a_3=0 \qquad (37)$$

at the times $t=-10,\ -5,\ 0$ and 6 respectively.

From Fig. 2, one can see that before a special interaction time, $t=0$, there are two solitoffs. One is

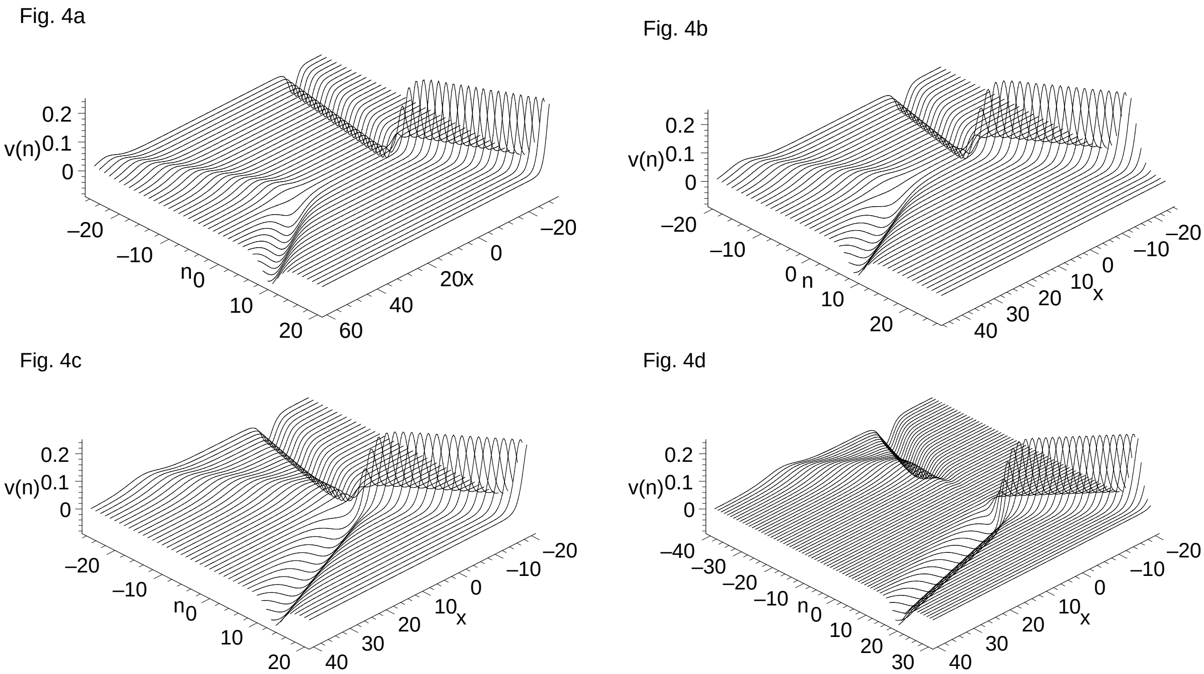

Fig. 4. An evolution plot of a particular semi-discrete four-resonant solitoff structure of the DDANNVE for the field $v(n)$ expressed by (27)–(36) and (39) at the times a: $t = -8$, b: $t = -4$, c: $t = 0$ and d: $t = 4$ respectively.

a straight line soliton with half-infinite length and the other solitoff possesses a finite length. As the time increases, the length of the finite-length solitoff becomes shorter and shorter. At the interaction time, only the half-infinite length solitoff survived. After that time, another finite length solitoff appears. As the time increases, the length of the new finite-length solitoff becomes longer and finally becomes another half-infinite long solitoff as $t \to \infty$.

Figure 3 reveals the evolution of four resonant solitoffs with variable length described, by (27) with (30), (31), (36), and

$$M = N = 2,\ N_1 = c_1 = -K_1 = -\omega_1 = \Omega_1 = \Omega_2 = 1,$$
$$-c_2 = K_2 = \frac{1}{3},\ a_0 = a_1 = 50,\ a_3 = 1,\ a_2 = \frac{1}{2} \qquad (38)$$

at the times $t = -8,\ -4,\ 0,\ 4$ and 28, respectively. Before a special interaction time, $t = 0$, four finite-length solitoffs are divided in two parts. In every part, two sollitoffs adhere each other. As the time increases, the length of the adhered solitoffs becomes shorter and shorter, while the amplitude becomes larger and larger. After that interaction time, two parts exchange one solitoff and become two new bounded pairs. Now, as the time increases, the solitoffs continuously develop their lengths in a different direction, while their amplitude becomes lower.

Figure 4 is a plot of a four infinitely long solitoff interaction solution shown by (27) with (30), (31), (36) and

$$M = N = 2,\ N_1 = c_1 = K_1 = -\omega_1 = \Omega_1 = \Omega_2 = 1,$$
$$-c_2 = K_2 = \frac{1}{3},\ a_0 = 1,\ a_1 = a_2 = 3,\ a_3 = 0 \qquad (39)$$

at the times $t = -5,\ -3,\ 0$ and 3, respectively. Similar to the first three cases, before the special interaction time, $t = 0$, four solitoffs are divided into two parts and every part contains two solitoffs and two solitoffs constitute a travelling bound pair. After that interaction time, two bounded pairs exchange one solitoff and move away in a different direction.

In Fig. 5, we plot a special evolution property caused by the resonant effects of six straight-line semi-discrete solitons expressed by (27) with (30)–(36) and

$$M = 3,\ N = 2,\ N_1 = c_1 = K_1 = \omega_1 = 1,$$
$$-c_2 = -4,\ K_2 = \tfrac{1}{3},\ K_3 = -3, \Omega_1 = \tfrac{3}{2},\ \Omega_2 = 0, \qquad (40)$$
$$\Omega_3 = 2,\ a_0 = 1,\ a_1 = a_2 = 3,\ a_3 = 0$$

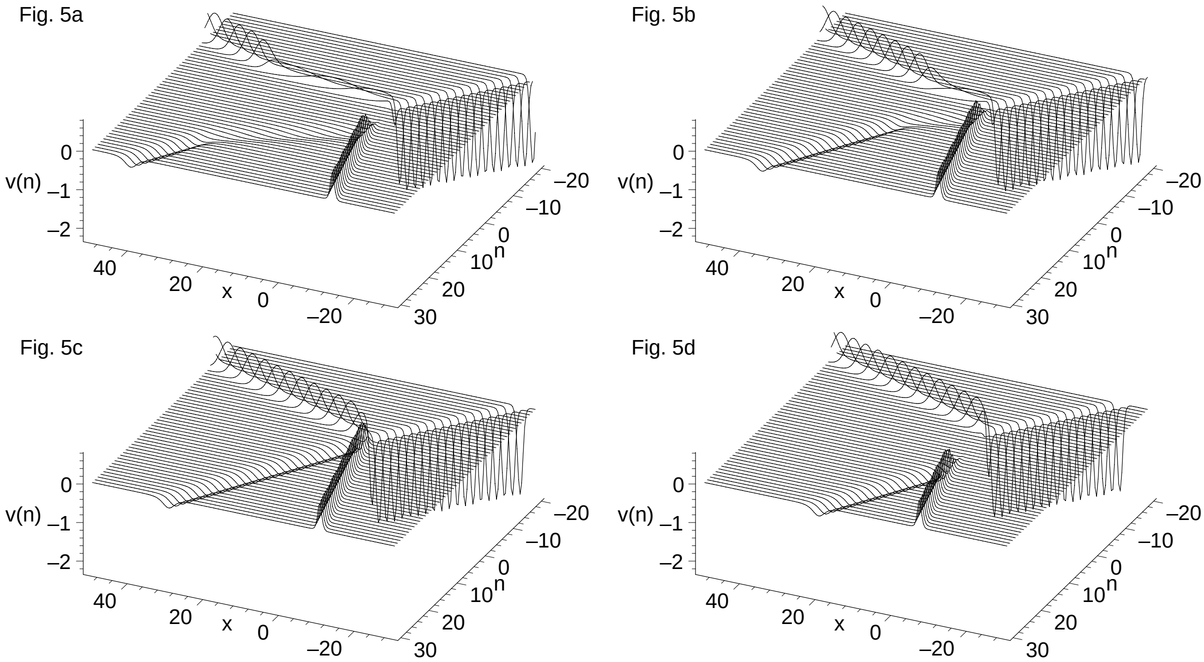

Fig. 5. An evolution plot of a particular semi-discrete six-resonant solitoff structure of the DDANNVE for the field v(n) expressed by (27)–(36) and (40) at the times a: $t = -10$, b: $t = -5$, c: $t = 0$ and d: $t = 10$ respectively.

at the times (a) $t = -10$, (b) $t = -5$, (c) $t = 0$ and (d) $t = 10$, respectively. Form Fig. 5 one can see that there are six solitoffs and four solitoffs respectively before and after the special interaction time, $t = 0$. Different from the situation of Fig. 4, after that interaction time, two solitoffs with finite lengths disappear and the other two pair of solitoffs move back away.

Example 2. Semi-discrete Oscillating Dromions and Lumps:

If some periodic functions in space variables are included in the functions $q(n)$ and p of (27), we may obtain some types of semi-discrete multi-dromion and multi-lump solutions with oscillating tails. The oscillating lump solution plotted in Fig. 2 is related to

$$p = \frac{1}{1+[(x+\omega t)(\cos(x+\omega t)+5/4)]^2}, \quad q(n) = \frac{1}{1+n^2}, \tag{41}$$

$$a_0 = a_3 = 1,\ a_1 = a_2 = 5 \tag{42}$$

at $t = 0$.

Because (41) possesses a travelling wave form, the evolution behavior of the oscillating lump plotted in

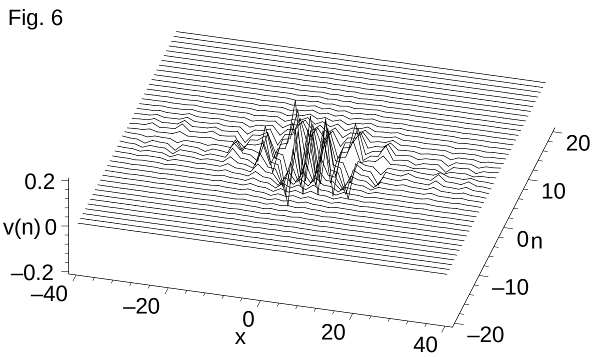

Fig. 6. Plot of a special oscillating lump solution of the DDANNVE for the quantity u expressed by (27) with (41) and (42) at $t = 0$.

Fig. 6 is trivial, it will move in the negative x direction with the velocity ω.

Example 3. Single ring soliton solution:

In high dimensions, in addition to the point-like ones, there may be some other types of physically significant localized excitations. Recently, we have found some different kinds of ring soliton solutions which are not identically equal to zero at some closed 2-dimen-

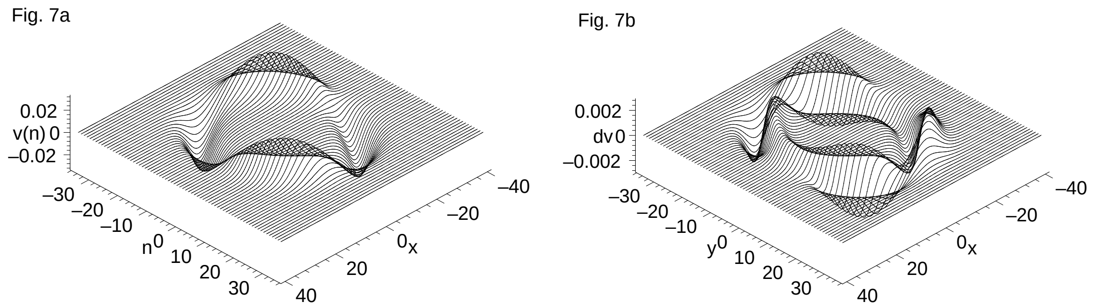

Fig. 7. a) Plot of a typical single saddle type semi-discrete ring soliton solution for the quantity $v(v)$ of the DDANNVE with the selections (43) and (44) at $t=0$. b) An error plot for the quantity dv defined by (45) under the difference step selection $\varepsilon = 1$.

sional and 3-dimensional curves and decay exponentially away from the curves [17], [25] – [29].

In Fig. 7a, a typical saddle type semi-discrete ring soliton solution is plotted for the quantity $v(n)$ with the selections

$$p = \exp\left(-\frac{(x+\omega t)^2}{80}+5\right),\ q(n) = \exp\left(\frac{n^2}{80}\right), \quad (43)$$

and

$$a_0 = a_3 = 0,\ a_1 = a_2 = 5, \quad (44)$$

at $t = 0$.

To show the error between the discrete model and the continuous model for the ANNV system, we plot the error quantity ($n\varepsilon = y$)

$$\mathrm{d}v \equiv \frac{v(n)}{\varepsilon} - v(y) \equiv \frac{-\mathrm{d}p_x(q(n\varepsilon+\varepsilon)-q(n\varepsilon))}{\varepsilon(a_0+a_1p+a_2q(n\varepsilon)+a_3pq(n\varepsilon))(a_0+a_1p+a_2q(n\varepsilon+\varepsilon)+a_3pq(n\varepsilon+\varepsilon))} + \frac{\mathrm{d}p_xq_y}{(a_0+a_1p+a_2q(y)+a_3pq(y))^2} \quad (45)$$

in Fig. 7b with the same function and parameter selections as in Fig. 7a and the difference step $\varepsilon = 1$. From Fig. 7b, one can find that the accuracy for the amplitude is about 10% under $\varepsilon = 1$. The further study we can find that for smaller ε the accuracy will be rapidly increased, for instance, $\sim 0.1\%$ for $\varepsilon = 0.1$ and $\sim 0.01\%$ for $\varepsilon = 0.01$.

In the continuous cases [11], it is known that there exist rich interaction properties among every type of localized excitations. The method to construct multiple continuous localized excitations with and without interaction properties has been analytically given in [15, 17, 30]. In this section, we extend the method to construct the multiple discrete coherent structures for both the DDANNVE and the SDDTE.

Similar to the continuous cases, one of the general ways to construct multiple localized 2+1 dimensional excitations is to select the functions p and $q(n)$ as multi-localized solitonic excitations with

$$p|_{t\to\mp\infty} = \sum_{i=1}^{M} p_i^{\mp},\ p_i^{\mp} \equiv p_i(x - c_it + \delta_i^{\mp}), \quad (46)$$

$$q(n)|_{t\to\mp\infty} = \sum_{j=1}^{M} q_j^{\mp}(n),\ q_{j,n}^{\mp} \equiv q_j(n - C_jt + \Delta_j^{\mp}), \quad (47)$$

where $\{p_i,\ q_j\}\ \forall\ i$ and j are localized functions. Under this selection, the potential U_1 expressed by (27), delivers $M\cdot N$ (2+1)-dimensional localized excitations with the asymptotic behavior

$$\begin{aligned} U_1|_{t\to\mp\infty} \quad &\to \quad \sum_{i,j} \frac{2dp_{ix}^{\mp}[q_j^{\mp}(n+1)-q_j^{\mp}(n)]}{[b_0^{\mp}+b_1^{\mp}p_i^{\mp}+b_2^{\mp}q_{j,n}^{\mp}+a_3p_i^{\mp}q_{j,n}^{\mp}][b_0^{\mp}+b_1^{\mp}p_i^{\mp}+b_2^{\mp}q_{j,n+1}^{\mp}+a_3p_i^{\mp}q_{j,n+1}^{\mp}]} \\ &\equiv \quad \sum_{i=1}^{M}\sum_{j=1}^{N} U_{ij}^{\mp}(x - c_it + \delta_i^{\mp},\ n - C_jt + \Delta_j^{\mp}) \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} U_{ij}^{\mp}, \end{aligned} \quad (48)$$

where

$$b_0^{\mp} = a_0 + a_1 P_i^{\mp} + a_2 Q_j^{\mp} + a_3 P_i^{\mp} Q_j^{\mp}, \tag{49}$$

$$b_1^{\mp} = a_1 + a_3 Q_j^{\mp},\ b_2^{\mp} = a_2 + a_3 P_i^{\mp}, \tag{50}$$

$$P_i^{\mp} = \sum_{j<i} p_j(\mp\infty) + \sum_{j>i} p_j(\pm\infty), \tag{51}$$

$$Q_j^{\mp} = \sum_{j<i} q_j(\mp\infty) + \sum_{j>i} q_j(\pm\infty). \tag{52}$$

In the above, it is assumed, without loss of generality, that $C_i > C_j,\ c_i > c_j$, if $i > j$.

Actually, there are some other types of multiple localized excitations by selecting the arbitrary functions in different ways. For instance, if we select the functions p^{-1} and $q(n)$ as multi-localized solitonic excitations with

$$\frac{1}{p}\bigg|_{t\to\mp\infty} = \sum_{i=1}^{M} p_i^{\mp},\ p_i^{\mp} \equiv p_i(x - c_i t + \delta_i^{\mp}), \tag{53}$$

$$q(n)|_{t\to\mp\infty} = \sum_{j=1}^{M} q_j^{\mp}(n),\ q_{j,n}^{\mp} \equiv q_j(n - C_j t + \Delta_j^{\mp}) \tag{54}$$

where $\{p_i,\ q_j\}\ \forall\ i$ and j are localized functions, then U_1 expressed by (27), delivers another type of $M \times N$ (2+1)-dimensional localized excitations with the asymptotic behavior

$$\begin{aligned} U_1|_{t\to\mp\infty} \quad &\to \quad \sum_{i,j} \frac{-2d p_{ix}^{\mp}[q_j^{\mp}(n+1) - q_j^{\mp}(n)]}{[c_0^{\mp} + c_1^{\mp} p_i^{\mp} + c_2^{\mp} q_{j,n}^{\mp} + a_2 p_i^{\mp} q_{j,n}^{\mp}][c_0^{\mp} + c_1^{\mp} p_i^{\mp} + c_2^{\mp} q_{j,n+1}^{\mp} + a_2 p_i^{\mp} q_{j,n+1}^{\mp}]} \\ &\equiv \quad \sum_{i=1}^{M}\sum_{j=1}^{N} U_{ij}^{\mp}(x - c_i t + \delta_i^{\mp},\ n - C_j t + \Delta_j^{\mp}) \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} U_{ij}^{\mp}, \end{aligned} \tag{55}$$

where

$$c_0^{\mp} = a_1 + a_0 P_i^{\mp} + a_3 Q_j^{\mp} + a_2 P_i^{\mp} Q_j^{\mp}, \tag{56}$$

$$c_1^{\mp} = a_0 + a_2 Q_j^{\mp},\ c_2^{\mp} = a_3 + a_2 P_i^{\mp}, \tag{57}$$

$P_i^{\mp}$ and $Q_j^{\mp}$ are expressed by (51) and (52).

From the expressions (48) and (55), we know that if $a_3 \neq 0$ for (48) and $a_2 \neq 0$ for (55), then the ijth localized excitation U_{ij} will be preserve its shape following the interaction iff (if and only if)

$$P_i^+ = P_i^-, \qquad Q_j^+ = Q_j^-. \tag{58}$$

The phase shifts of the ijth localized excitation U_{ij} read

$$\delta_i^+ - \delta_i^- \tag{59}$$

in the x direction and

$$\Delta_j^+ - \Delta_j^-. \tag{60}$$

in the n direction.

In the continuous cases, multiple localized excitations similar to the first type of selections like (46) and (47) have been shown in [17]. Here we just give a special example related to the second type of the selections.

Example 4. Two-ring soliton solution with completely elastic interaction behavior.

In Fig. 8, a typical semi-discrete two-ring soliton solution is plotted for the quantity $v(n)$ with the selections

$$\begin{aligned} q^{-1} &= \exp\Big(\frac{-(x-10t)^2}{40} + 20\Big) \\ &\quad + \exp\Big(\frac{-(x+10t)^2}{50} + 25\Big), \\ q(n) &= \Big(-\frac{n^2}{10} + 5\Big), \end{aligned} \tag{61}$$

and

$$a_0 = a_3 = 0,\ a_1 = a_2 = 1. \tag{62}$$

From Figs. 8a to 8f, it is seen that the interaction between two ring solitons is completely elastic. Actually, from the analytic discussions from (53) to (60) and the concrete selections (61) with (62), we can also know that the interaction is completely elastic because (58) is satisfied ($P_1^+ = P_1^- = 0,\ P_2^+ = P_2^- = 0,\ Q_1^+ = Q_1^-$). Furthermore, there is no phase shift for both ring solitons because $\delta_1^+ = \delta_1^- = \delta_2^+ = \delta_2^- = \Delta_1^+ = \Delta_1^- = 0$ for the selections (61). To find multiple ring soliton solutions with phase shifts, one has to use much more complicated selections similar to the continuous cases [27].

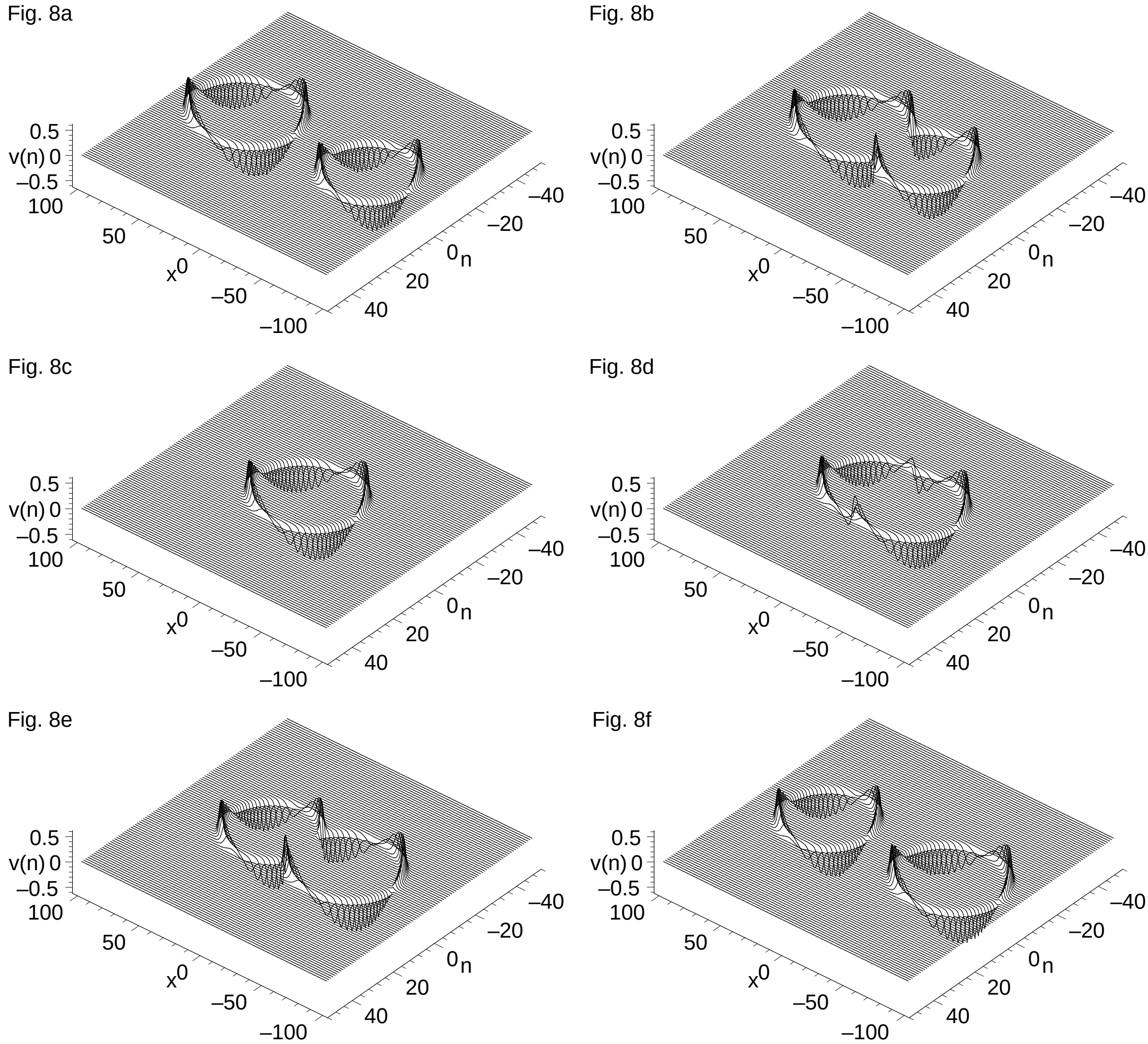

Fig. 8. Plot of a typical semi-discrete two-ring soliton solution with completely elastic interaction for the quantity u of the DDANNVE with the selections (61) and (62) at the times a: $t=-5$, b: $t=-3$, c: $t=0$, d: $t=1.5$, e: $t=3$, f: $t=5$.

5. Conclusions and Discussions

In the previous studies [9 – 17], we have successfully obtained rich classes of exact solutions for some famous (2+1)-dimensional nonlinear continuous integrable models and a special differential-difference Toda equation via the MLVSA. In the present paper, we have further applied the BLVSA to find some kinds of exact solutions of the DDANNVE.

In continuous cases, a quite universal formula has been found to describe some types of special solutions of suitable fields or potentials of the MLVSA solvable models. For the DDANNVE, a semi-discrete form (which is not completely the same as that of the SD-DTE) of the formula is obtained for a suitable quantity. An arbitrary function, related to the continuous space variable, is introduced in the semi-discrete form (27). Another function, included in the formula, is an arbitrary solution of the Riccati equation with respect to the time t, and the Riccati equation is totally the same as that in some continuous MLVSA solvable models, though the space variable is replaced by a discrete one.

By selecting the arbitrary functions appropriately, one can construct abundant semi-discrete localized excitations like the multiple solitoffs, dromions, lumps, breathers, instantons and ring soliton solutions. The semi-discrete localized solutions are quite similar to those of continuous cases shown in [11] and those of the SDDTE.

Especially, two possible ways to construct multiple localized excitations with and without completely elastic interaction properties are proposed and a special selection of a saddle type discrete two-ring soliton solution is depicted.

In the continuous cases, though the MLVSA has been applied to various systems, some other important integrable systems, especially the Kadomtsev-Petviashvili equation, the Sawada-Kortera equation and the usual (2+1)-dimensional Toda system [31] have not yet been solved. So, more about the method should be studied.

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